Anisotropy and inflation in Bianchi I brane worlds

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Abstract

After a more general assumption on the influence of the bulk on the brane, we extend some conclusions by Maartens *et al.* [1] and Santos *et al.* [2] on the asymptotic behavior of Bianchi I brane worlds. As a consequence of the nonlocal anisotropic stresses induced by the bulk, in most of our models, the brane does not isotropize and the nonlocal energy does not vanish in the limit in which the mean radius goes to infinity. We have also found the intriguing possibility that the inflation due to the cosmological constant might be prevented by the interaction with the bulk. We show that the problem for the mean radius can be completely solved in our models, which include as particular cases those in the references above.

I. INTRODUCTION

The main stream approach to try and solve the problems arising from the break down of Einstein's theory at high energies is that of considering it a particular limit of a more general theory, as happened with Newton's theory in relation to that of Einstein's. One of this schemes is the brane gravity picture [3, 4], according to which matter fields are confined to a hypersurface (the brane) with three spatial dimensions embedded in a higher dimensional space (the bulk) on which gravity can act also.

Considerable efforts in this area of research have been directed to testing that scheme by deducing cosmological implications from it. The usual framework for these studies is the geometric approach of [5], which provides the gravitational field equations induced on the brane along with conservation equations for bulk degrees of freedom. Reviews on this topic can be found, for instance, in [6, 7, 8].

Studies concerning FRW universes [9] hinted that the dynamics of the early Universe in this new scenario is rather peculiar because of the modifications in the Friedmann equation. Then, when anisotropic models were considered [10], it was shown that discrepancies with respect to the standard scenario extended to the shear dynamics as well.

In the set-up of general relativity, Wald [11] showed that a positive cosmological constant leads to the isotropization of expanding homogeneous cosmological models . The anisotropy dissipation in brane world inflation in the absence of effective cosmological constant was analyzed by Maartens, Sahni and Saini [1]. Taking into account the cosmological constant, Santos, Vernizzi and Ferreira derived a set of sufficient conditions that allow extending Wald's result to the brane world scenario [2] (see also [12] for an alternative view on the same problem). The last three works were done directly on the brane, where an additional hypothesis has to be done for the nonlocal anisotropic stresses induced by the bulk, for they are not given by any evolution equation on the brane. This naturally raises the question of to what extent the results depend on the aforementioned hypothesis.

The goal of this work is to extend the results in [1, 2] by considering a more general additional hypothesis. We will analyse, as done in those works, a Bianchi I brane embedded in a 5-dimensional bulk. We assume that the nonlocal stresses satisfy a condition which reduces to the one considered in Refs. [1, 2] in particular cases. We will show that the evolution equations on the brane can be integrated and conclude that, generically, we have

exponential inflation or an asymptotic power law for the mean radius, but the models do not isotropize unless they belong to the class considered in the aforementioned works.

II. BIANCHI I BRANE MODELS

Let us consider a Bianchi I brane with the induced metric

$$ds^2 = -dt^2 + a_1^2(t)\, dx^2 + a_2^2(t)\, dy^2 + a_3^2(t)\, dz^2 \tag{1}$$

when the matter on the brane is a perfect fluid of density ρ and pressure p. We define as usual the mean radius $a \equiv (a_1 a_2 a_3)^{1/3}$ and the mean Hubble parameter $H \equiv \dot{a}/a = \frac{1}{3}\sum_i H_i$, with $H_i = \dot{a}_i/a_i$.

We will be using the equations induced on the brane derived by Shiromizu, Maeda and Sasaki [5] but follow the notation of [1, 2]. For the metric (1), the dynamics of the mean radius a in this model can be described by the following equations on the brane:

$$3H^2 = \frac{6}{\kappa^2\lambda}\mathcal{U} + \frac{1}{2}\sigma^2 + \Lambda + \kappa^2\rho\left(1 + \frac{\rho}{2\lambda}\right), \tag{2}$$

$$3\dot{H} + 3H^2 + \sigma^2 + \frac{6}{\kappa^2\lambda}\mathcal{U} = \Lambda - \frac{\kappa^2}{2}(\rho + 3p) - \frac{\kappa^2}{2}(2\rho + 3p)\frac{\rho}{\lambda}, \tag{3}$$

$$\dot{\sigma}_{\mu\nu} + 3H\sigma_{\mu\nu} = \frac{6}{\kappa^2\lambda}\mathcal{P}_{\mu\nu}, \tag{4}$$

$$\dot{\mathcal{U}} + 4H\mathcal{U} + \sigma^{\mu\nu}\mathcal{P}_{\mu\nu} = 0,, \tag{5}$$

where a dot denotes $u^\mu\nabla_\mu$, with $u^\mu = \partial/\partial t$. Throughout the paper $\kappa^2/8\pi$ and $\tilde{\kappa}^2/8\pi$ will, respectively, be the effective Newton constant on the brane and in the bulk ; Λ will be the effective 4D cosmological constant, and the tension on the brane will be $\lambda \equiv 6\kappa^2/\tilde{\kappa}^4$. On the other hand, $\mathcal{U}$ stands for the effective nonlocal energy density, and $\sigma_{\mu\nu}$ denotes the shear scalar, which satisfies (4) and

$$\sigma^2 \equiv \sigma^{\mu\nu}\sigma_{\mu\nu} = \sum_{i=1}^{3}(H_i - H)^2. \tag{6}$$

In addition, if u^μ is the four velocity of an observer on the brane comoving with matter, then $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ projects into the comoving rest-space, and $\mathcal{U}$ and $\mathcal{P}_{\mu\nu}$ will be related to $\mathcal{E}_{\mu\nu}$, which is the projection on the brane of the 5D Weyl tensor, through

$$\mathcal{U} = -\frac{\kappa^2\lambda}{6}\mathcal{E}_{\mu\nu}u^\mu u^\nu, \tag{7}$$

$$\mathcal{P}_{\mu\nu} = -\frac{\kappa^2\lambda}{6}\left(h_\mu^{\alpha}h_\nu^{\beta} - \frac{1}{2}h^{\alpha\beta}h_{\mu\nu}\right)\mathcal{E}_{\alpha\beta}. \tag{8}$$

There is also, in principle, the constraint that the projected spatial covariant derivative of $\mathcal{P}_{\mu\nu}$ vanishes:

$$\mathrm{D}^{\nu}\mathcal{P}_{\mu\nu} = 0, \tag{9}$$

but this result holds identically for the metric (1). On the other hand, as a consequence of system (2)–(5), we get the conservation law

$$\dot{\rho} + 3H(\rho + p) = 0. \tag{10}$$

III. EXACT EXAMPLES AND ASYMPTOTIC BEHAVIOR

Since there is no evolution equation on the brane for the nonlocal anisotropic stress $\mathcal{P}_{\mu\nu}$, but only the constraint (9), which in this case does not provide any information, some additional hypothesis is necessary to integrate the system (2)–(5).

In order to get some insight on the problem, we will restrict ourselves for a moment to the vacuum case $\rho = p = 0$, where we get from (2)–(3)

$$\mathcal{U} = \frac{\kappa^2\lambda}{2}\left(\dot{H} + 3H^2 - \Lambda\right). \tag{11}$$

Let us consider under which conditions the stable asymptotic behavior may approach a power law $a \sim t^k$, $(k > 0)$. In such a case $\dot{H}, H \to 0$, $\mathcal{U} \to -\kappa^2\lambda\Lambda/2$ and, because of (5),

$$\sigma^{\mu\nu}\mathcal{P}_{\mu\nu} \to 2\kappa^2\lambda\Lambda H. \tag{12}$$

We will show in the following that, if one chooses the unknown $\mathcal{P}_{\mu\nu}$ so that this condition is satisfied, the asymptotic behavior described by a power law is stable. Furthermore, this will happen even in the presence of a fluid.

So as to integrate the system (2)–(5) one often [1] assumes $\mathcal{U} = 0$ or the more general $\sigma^{\mu\nu}\mathcal{P}_{\mu\nu} = 0$, which has been used in [2] to discuss the stability of the de Sitter spacetime. It has been proved [13] that spatial homogeneity follows from the integrability conditions for vanishing non-local anisotropic stress and energy flux, which are two of the three bulk degrees of freedom. Therefore, the considerable simplification arising from switching off those quantities is consistent with having a Bianchi I metric on the brane.

It must be pointed out that other choices can be found in the literature; Barrow and Maartens [14], in an investigation of early times shear anisotropy in an inhomogeneous

universe, assumed that $\mathcal{P}_{\mu\nu}$ behaves qualitatively like a general 4D anisotropic stress, in particular they chose $\mathcal{P}_{\mu\nu}$ to be proportional to the energy density of the anisotropic source (which we will denote with $\tilde{\rho}$) so that $\mathcal{P}_{\mu\nu} = \tilde{\rho}\, C_{\mu\nu}$ with $\dot{C}_{\mu\nu} = 0$, $\sqrt{C_{\mu\nu}C^{\mu\nu}} = \mathcal{O}(1)$, and $\tilde{\rho} \ll \rho$.

Although the asymptotic behavior (12) may happen with many choices of $\sigma^{\mu\nu}\mathcal{P}_{\mu\nu}$, we will consider only a simple family of models. In the following, we will assume that $\sigma^{\mu\nu}\mathcal{P}_{\mu\nu}$ is proportional to the Hubble parameter, so that it can be written as

$$\sigma^{\mu\nu}\mathcal{P}_{\mu\nu} = 2\kappa^2\lambda\Gamma(a)\, H, \tag{13}$$

which reduces to the cases discussed in [1, 2] when the function $\Gamma(a)$ vanishes. By making this hypothesis, we use a less restrictive assumption to check whether the conclusions reached in [1, 2] hold in a more general context, while still being able to integrate the evolution equations. Note that (12) is recovered, even before the asymptotic behavior is reached, provided $\Gamma(a) = \Lambda$.

Exactly as happens with the restrictive $\Gamma = 0$ case often used in the bibliography, it remains an open problem whether the more general assumption we make is compatible with a full 5D solution (see [15] for a recent attempt at tackling the 5D problem). Judging on the suitability of such guesses is, therefore, not possible for the time being, but they definitely help unveiling possible unexpected consequences of the interference between bulk and brane.

The nonlocal energy density $\mathcal{U}$ is given by the conservation equation (5), which is easily solved under the assumption (13):

$$\mathcal{U} = \frac{\kappa^2\lambda u_0}{a^4} - \frac{2\kappa^2\lambda}{a^4}\int a^3\Gamma(a)\, da, \tag{14}$$

where u_0 is an arbitrary integration constant. We see here that, for expanding universes,

$$\lim_{a\to\infty}\mathcal{U} = -\frac{1}{2}\kappa^2\lambda \lim_{a\to\infty}\Gamma(a), \tag{15}$$

(provided the limit on the right side exists or is infinite) so that it does not always vanish asymptotically, although it will go to zero in the special cases in which $\Gamma(a) \to 0$ (which of course include the choice $\Gamma(a) = 0$ of Refs. [1, 2]), where it reduces to the result by Toporensky [16]. Expression (15) suggests there is the possibility that, as a consequence of the interaction of the brane with the bulk through $\mathcal{P}_{\mu\nu}$, the asymptotic vanishing of $\mathcal{U}$ arising in the cases studied in [2] might be evitable.

By contracting (4) with $\sigma^{\mu\nu}$ and using (13), one readily gets

$$\sigma^2 = \frac{\sigma_0^2}{a^6} + \frac{24}{a^6}\int a^5\Gamma(a)\,da, \tag{16}$$

for any constant σ_0, so that

$$\lim_{a\to\infty}\sigma^2 = 4\lim_{a\to\infty}\Gamma(a). \tag{17}$$

Then, it is clear that asymptotically Γ must be non-negative. If $\Gamma \to 0$ the brane world isotropizes as $a \to \infty$, as described in [1, 2], but, for all other asymptotic behaviours of Γ, there will be a remaining anisotropy induced by the nonlocal anisotropic stresses. Note that there is also the possibility that the anisotropy grows with a, which would correspond to $\mathcal{U} < 0$ at late times. Models which do not isotropize in a recollapsing situation, unlike ours, were found in [17], and they were characterized by $\mathcal{P}_{\mu\nu}$ and $\mathcal{U} < 0$.

Clearly, our work and many others suggest that the interplay between the bulk degrees of freedom and the dynamics on the brane is rather non-trivial. An interesting illustration of this is a recent work [18] which investigated the dynamics of a flat isotropic brane world with a perfect fluid with equation of state $p = (\gamma - 1)\rho$ and a scalar field with a power-law potential. There it was found that the number and the stability of fixed points of the system describing the dynamics of the model would depend not only on whether $\mathcal{U}$ vanishes or not, but also on its sign.

If we assume the equation of state $p = (\gamma - 1)\rho$, the conservation law (10) is equivalent to

$$\rho = \frac{\rho_0}{\kappa^2 a^{3\gamma}}, \tag{18}$$

with an arbitrary constant ρ_0.

If we insert (14), (16) and (18) in (2), we get the generalized Friedmann equation

$$3H^2 = \frac{6u_0}{a^4} + \frac{\sigma_0^2}{2a^6} + \frac{\rho_0}{a^{3\gamma}} + \frac{\rho_0^2}{2\kappa^2\lambda a^{6\gamma}} + \Lambda - \frac{24}{a^6}\int\left[a\int a^3\Gamma(a)\,da\right]da. \tag{19}$$

The orbits $(H(\mathcal{U}), \sigma(\mathcal{U}))$ were found in [2] for the special case $\Gamma = 0$, but we can see that, for any $\Gamma(a)$, the problem for $a(t)$, $\sigma(t)$ and $\mathcal{U}(t)$ may be completely solved from equation (19) by means of a quadrature and a function inversion. The integrals may be explicitly computed (in terms of elementary or elliptic functions) for different choices for the constants u_0, σ_0, ρ_0, γ and Λ and the function $\Gamma(a)$. The simplest cases are solutions with no fluid

($\rho_0 = 0$). If $\Gamma(a) = \Lambda$, by using the parameter $0 < w < \infty$ one may write the solution as

$$t = \frac{\sigma_0^2}{48\sqrt{2u_0^3}} (\sinh w - w) , \tag{20}$$

$$a = \frac{\sigma_0}{2\sqrt{3u_0}} \sinh \frac{w}{2}, \tag{21}$$

which at late time corresponds to $a \sim t^{1/2}$.

In contrast, if $u_0 = \rho_0 = 0$ and $\Gamma(a) = \alpha$ for a constant $\alpha < \Lambda$, we get

$$a^6 = \frac{\sigma_0^2}{2(\Lambda - \alpha)} \sinh^2 \sqrt{3(\Lambda - \alpha)}\, t. \tag{22}$$

Other exact solutions with $u_0 = 0$ can obtained with dust ($\gamma = 1$) or a stiff fluid ($\gamma = 2$). Let us look first at the dust cases, which, arguably, are very interesting from the observational point of view [19]. For $\Gamma(a) = \alpha$ we have

$$a^3 = \frac{\rho_0}{\Lambda - \alpha} \sinh^2 \frac{\sqrt{3(\Lambda - \alpha)}\, t}{2} + \sqrt{\frac{\rho_0^2 + \kappa^2 \lambda \sigma_0^2}{2\kappa^2 \lambda (\Lambda - \alpha)}} \sinh \sqrt{3(\Lambda - \alpha)}\, t. \tag{23}$$

The latter is a generalization of the Heckmann-Shucking metric [20, 21] that has not been discussed in the literature so far. Clearly, the $\alpha \to \Lambda$ limit of the solution (23) is regular, with the form of a second-order polynomial in t. The effects of the shear and the quadratic corrections are of the same order, they dominate at early times, and $a \sim t^{1/3}$. In contrast, at late times the model is neither aware of the anisotropy nor of the extra dimensional ingredients, and $\log a \propto t$, or $a \sim t^{2/3}$ if $\Lambda = \alpha$. On the other hand, for a stiff fluid ($\gamma = 2$), and $\Gamma(a) = \alpha$, we have

$$a^6 = \frac{2\,\rho_0 + \sigma_0^2}{2(\Lambda - \alpha)} \sinh^2 \sqrt{3(\Lambda - \alpha)}\, t + \frac{\rho_0}{\sqrt{2\,\kappa^2\, \lambda (\Lambda - \alpha)}} \sinh 2\sqrt{3(\Lambda - \alpha)}\, t. \tag{24}$$

This solution too has a regular $\alpha \to \Lambda$ limit in the form of a second-order polynomial in t. Like in the dust solution above, the effects of the shear and the quadratic corrections are equally important, but they dominate at late times, instead; in that regime we either have $\log a \propto t$, or $a \sim t^{1/3}$ if $\Lambda = \alpha$. On the contrary, at early times the model becomes isotropic and relativistic, and $a \sim t^{1/6}$.

We can also obtain directly from (19) some general late-time results of interest. If asymptotically $\Gamma(a) \sim \alpha$ is constant, then for $\Lambda - \alpha > 0$, the brane inflates exponentially when $a \to \infty$, as happened for $\alpha = 0$, $\Lambda > 0$ [2]. But now there is another interesting possibility:

the repulsion due to the cosmological constant may be neutralized by the bulk influence in which case the asymptotic behavior of a is a power law. For simplicity we show this in vacuum and write (11) in terms of the variable $\mu = -2\dot{H}/3H^2$, which becomes a constant μ_0 for the power-law solutions $a \propto t^{2/3\mu_0}$, so

$$\dot{\mu} + \left[3H\mu - \frac{8H\mathcal{U} + 2\sigma^{\mu\nu}P_{\mu\nu}}{2\mathcal{U} + \kappa^2\lambda\Lambda}\right](2-\mu) = 0. \tag{25}$$

The existence of the constant solution, $\mu = \mu_0$, in the last equation requires that

$$\sigma^{\mu\nu}P_{\mu\nu} = 3H\left[\mu_0\left(\mathcal{U} + \frac{\kappa^2\lambda\Lambda}{2}\right) - \frac{4}{3}\mathcal{U}\right], \tag{26}$$

this means that, for any solution of (11) which asymptotically approaches to a power law, then the quantity $\sigma^{\mu\nu}P_{\mu\nu}$ asymptotically behaves as (26). Now, inserting (26) in (5) and (4), we get the asymptotic expression for the effective nonlocal energy density

$$\mathcal{U} = \frac{\kappa^2\lambda\alpha}{a^{3\mu_0}} - \frac{1}{2}\kappa^2\lambda\Lambda, \tag{27}$$

and the shear scalar

$$\sigma^2 = \frac{\sigma_0^2}{a^6} + \frac{12\alpha(\mu_0 - 4/3)}{(2-\mu_0)a^{3\mu_0}} + 4\Lambda, \tag{28}$$

where α and σ_0^2 are arbitrary constants. In addition, introducing (26) into (25), it reads as

$$\dot{\mu} = -3H(\mu - \mu_0)(2-\mu). \tag{29}$$

It is easy to see that, for ordinary fluids, $\mu_0 < 2$, the solutions of Eq. (29) converge, respectively, to the stable fixed point $\mu = \mu_0$, which describes the asymptotic power-law solutions $a = t^{2/3\mu_0}$. For instance, if we additionally assume a perfect fluid source, after using (18), (27) and (28), the Einstein equation (2) becomes

$$3H^2 = \frac{6\alpha}{a^{3\mu_0}} + \frac{\sigma_0^2}{2a^6} + \frac{4\alpha}{(2-\mu_0)a^{3\mu_0}} + \frac{\rho_0}{a^{3\gamma}} + \frac{\rho_0^2}{2\kappa^2\lambda a^{6\gamma}}. \tag{30}$$

Hence, for $\gamma > \mu_0$, asymptotically the brane will remain anisotropic ($\sigma^2 \sim 4\Lambda$) and expand as $a \sim t^{2/3\mu_0}$ or expand as $a \sim t^{2/3\gamma}$ for $\gamma < \mu_0$. Note that, the exact solution given by (20) and (21) for a model with no fluid, is also the approximated late time solution when

$\mu_0 = 4/3 \leq \gamma$ or $\gamma = 4/3 \leq \mu_0$. In both cases asymptotically the scale factor behaves a $a \sim t^{1/2}$.

In (26) we assumed that $\sigma^{\mu\nu} P_{\mu\nu}$ depends on a and $\mathcal{U}$. We could instead assume that it depends only on a so that asymptotically (13) holds with

$$\Gamma(a) = \Lambda + \frac{\beta}{a^{3\delta}}, \tag{31}$$

for some constants β and $\delta > 0$. It is easy to see that in this case also the asymptotic behavior of a is a power law. For instance, if we additionally assume that $\gamma > 4/3$ (or that there is no fluid), asymptotically the brane will remain anisotropic ($\sigma^2 \sim 4\Lambda$) and expand as $a \sim t^{2/3\delta}$ for $\delta < 4/3$ and as $a \sim t^{1/2}$ for $\delta \geq 4/3$ (or $\beta = 0$). For $\gamma < 4/3$, one would have $a \sim t^{2/3\nu}$, with $\nu = \min(\gamma, \delta)$.

IV. CONCLUSIONS

By making the more general assumption (13) on the (unknown) influence of the bulk on the brane, we have shown that some conclusions on the asymptotic behavior of Bianchi I brane worlds in Refs. [1, 2] can be generalized. Due to the nonlocal stresses, in most of our models, the nonlocal energy does not vanish in the limit $a \to \infty$, and the brane does not isotropize. We have also found that, although nearly all our models inflate, there also exist the possibility that the inflation due to the cosmological constant might be prevented by the interaction with the bulk. Finally, we have shown that the problem for the mean radius a (as well as for σ^2 and $\mathcal{U}$) can be completely solved in our models, which include as particular cases the Bianchi I branes for which the orbits and stability were analyzed in [2].

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